Name: ____________________

Instructor: ____________________

Math 10560, Practice Exam 2.
March 19, 2025

- The Honor Code is in effect for this examination. All work is to be your own.
- No calculators.
- The exam lasts for 1 hour and 15 min.
- Be sure that your name is on every page in case pages become detached.
- Be sure that you have all 15 pages of the test.
- Each multiple choice question is worth 7 points. Your score will be the sum of the best 10 scores on the multiple choice questions plus your score on questions 13-16.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	(a)	(b)	(c)	(d)	(e)
2.	(a)	(b)	(c)	(d)	(e)
3.	(a)	(b)	(c)	(d)	(e)
4.	(a)	(b)	(c)	(d)	(e)
5.	(a)	(b)	(c)	(d)	(e)
6.	(a)	(b)	(c)	(d)	(e)
7.	(a)	(b)	(c)	(d)	(e)
8.	(a)	(b)	(c)	(d)	(e)
9.	(a)	(b)	(c)	(d)	(e)
10.	(a)	(b)	(c)	(d)	(e)
11.	(a)	(b)	(c)	(d)	(e)
12.	(a)	(b)	(c)	(d)	(e)

Please do NOT write in this box.

Multiple Choice	________
13.	________
14.	________
15.	________
Total	________

Name: ______________________________

Instructor: ___________________________

Multiple Choice

1.(7 pts.) Use Simpson's rule with $n = 4$ to appoximate the integral $\int_0^4 f(x)dx$ where a table of values for the function $f(x)$ is given below.

x	0	1	2	3	4
$f(x)$	2	1	2	3	5

$\int_0^4 f(x)\,dx \approx \frac{\Delta x}{3}\left[f(0) + 4\cdot f(1) + 2\cdot f(2) + 4\cdot f(3) + f(4)\right] = \frac{1}{3}\left[2 + 4 + 4 + 12 + 5\right] = \frac{27}{3} = 9.$

(a) 11 (b) 8 (c) 9 (d) 9.5 (e) 10.4

2.(7 pts.) Evaluate the improper integral

$$\int_4^\infty \frac{1}{(x-2)(x-3)}\,dx.$$

Using a partial fraction expansion $\int \frac{1}{(x-2)(x-3)}dx = \ln|\frac{x-3}{x-2}| + C.$

Therefore $\int_4^\infty \frac{1}{(x-2)(x-3)}dx = \lim_{t\to\infty} \ln|\frac{t-3}{t-2}| - \ln|\frac{1}{2}| = 0 + \ln 2.$

(a) the integral diverges (b) $\ln 3$ (c) $\ln \frac{1}{2}$

(d) $\ln 2$ (e) $3\ln 2$

Name: ____________________________

Instructor: _________________________

3.(7 pts.) What can be said about the integrals

$$(i)\int_0^1 \frac{e^x}{x^2}\,dx;$$
$$(ii)\int_1^\infty \frac{\cos^2 x}{x^2}\,dx?$$

Integral (i) diverges by the Comparison Theorem since the integrand is greater than $\frac{1}{x^2}$.

Integral (ii) converges by the Comparison Theorem since the integrand is less than $\frac{1}{x^2}$.

(a) (i) converges and (ii) diverges

(b) both (i) and (ii) diverge

(c) (i) diverges and (ii) converges

(d) neither integral (i) nor (ii) is improper

(e) both (i) and (ii) converge

4.(7 pts.) Which of the following is an expression for the arclength of the curve $y = \cos x$ between $x = \frac{-\pi}{2}$ and $x = \frac{\pi}{2}$?

The arclength formula gives the answer as $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 + (-sinx)^2}dx$.

Name: ______________________________

Instructor: __________________________

(a) $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\cos^2 x}\, dx.$

(b) $2\int_{0}^{\frac{\pi}{2}} \sqrt{1+2\sin^2 x}\, dx.$

(c) $\frac{\pi^2}{2}$

(d) $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\sin^2 x}\, dx.$

(e) $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sqrt{1+\sin^2 x}\, dx.$

Name: ____________________

Instructor: ____________________

5.(7 pts.) Consider the following sequences:

$$(I)\ \left\{(-1)^n \frac{n^2-1}{2^n}\right\}_{n=1}^{\infty} \qquad (II)\ \left\{(-1)^n \frac{n^2-1}{2n^2}\right\}_{n=1}^{\infty} \qquad (III)\ \left\{(-1)^n n\ln(n)\right\}_{n=1}^{\infty}$$

Which of the following statements is true?

(I): By applying L'Hospital's Rule to the function $f(x) = \dfrac{x^2-1}{2^x}$ we can see that $\lim_{x\to\infty} f(x) = 0$. Thus $\lim_{n\to\infty} \dfrac{n^2-1}{2^n} = 0$. But for $n \geq 1$,

$$\frac{n^2-1}{2^n} = \left|(-1)^n \frac{n^2-1}{2^n}\right|,$$

so the sequence (I) also converges to 0.

(II): $\lim_{n\to\infty} \dfrac{n^2-1}{2n^2} = 1/2$, so as n grows large, the expression $(-1)^n \dfrac{n^2-1}{2n^2}$ oscillates between values close to $+1/2$ (when n is even) and values close to $-1/2$ (when n is odd). Thus the sequence (II) diverges.

(III): As $n \to \infty$, $n\ln(n)$ grows arbitrarily large. The factor of $(-1)^n$ in sequence (III) makes the values oscillate between positive values of large magnitude and negative values of large magnitude. Thus the sequence (III) diverges.

(a) Sequences II and III converge but sequence I diverges.

(b) Sequence I converges but sequences II and III diverge.

(c) All three sequences diverge.

(d) Sequences I and II converge but sequence III diverges.

(e) All three sequences converge.

6.(7 pts.) Find the sum of the following series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n 2^{n+1}}{3^n}.$$

This is a geometric series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots = \begin{cases} \text{converges to} & \dfrac{a}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & & \text{if } |r| \geq 1. \end{cases}$$

Name: ____________________________

Instructor: __________________________

(technically we should check if a_{n+1}/a_n is a constant r in order to check this.) We can identify a by calculating the first term with a_1. When $n = 1$, we get

$$a = a_1 = \frac{(-1)^1 2^{1+1}}{3^1} = -\frac{2^2}{3}.$$

When $n = 2$, we get

$$ar = a_2 = \frac{(-1)^2 2^{2+1}}{3^2} = \frac{2^3}{3^2}.$$

Now we have

$$r = \frac{a_2}{a_1} = \left(\frac{2^3}{3^2}\right)/\left(-\frac{2^2}{3}\right) = -\left(\frac{2^3}{3^2}\right)\left(\frac{3}{2^2}\right) = -\frac{2}{3}.$$

This means $a = -\frac{4}{3}$ and $r = -\frac{2}{3}$. Then $|r| < 1$ so the series converges to

$$\frac{a}{1-r} = \frac{-\frac{4}{3}}{1 - \frac{-2}{3}} = -\frac{4}{5}$$

(a) This series diverges. (b) $-\frac{4}{5}$ (c) $\frac{4}{5}$

(d) $-\frac{3}{5}$ (e) $\frac{3}{5}$

Name: ______________________________

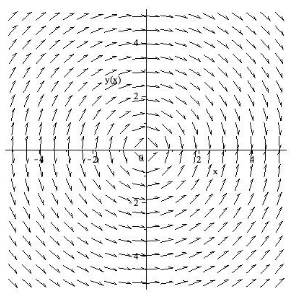
Instructor: __________________________

7.(7 pts.) Which of the following gives the direction field for the differential equation

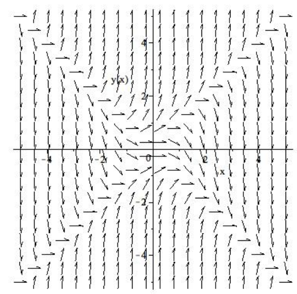
$$y' = y^2 - x^2$$

Note the letter corresponding to each graph is at the lower left of the graph.

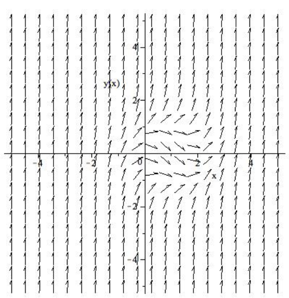
For points on the line $y = x$, we must have $y' = 0$. Also for points on the line $y = -x$, we must have $y' = 0$. Hence along both diagonals of the plane, we must have $y' = 0$ and the answer must be (b).



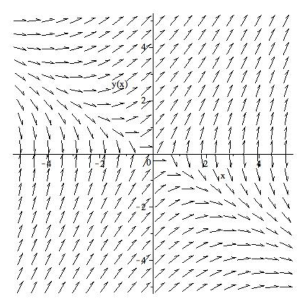
(a)



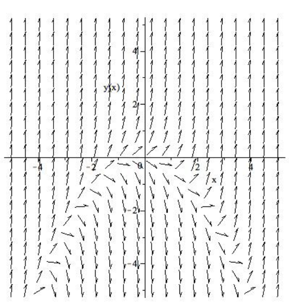
(b)



(c)



(d)



(e)

Name: ______________________________

Instructor: __________________________

8.(7 pts.) Use Euler's method with step size 0.1 to estimate $y(1.2)$ where $y(x)$ is the solution to the initial value problem

$$y' = xy + 1 \qquad y(1) = 0.$$

$$x_0 = 1, \quad y_0 = 0$$

$$x_1 = x_1 + h = 1.1, \quad y_1 = y_0 + h(x_0 y_0 + 1) = 0 + (0.1)(1 \cdot 0 + 1) = 0.1$$

$$x_2 = x_1 + h = 1.2, \quad y_2 = y_1 + h(x_1 y_1 + 1) = 0.1 + (0.1)((1.1)(0.1) + 1)$$

$$= 0.1 + 0.1(0.11 + 1) = 0.1 + 0.1(1.11) = 0.1 + 0.111 = 0.211$$

(a) $y(1.2) \approx .101$ (b) $y(1.2) \approx .112$ (c) $y(1.2) \approx .211$

(d) $y(1.2) \approx .201$ (e) $y(1.2) \approx .111$

Name: ______________________

Instructor: ______________________

9.(7 pts.) Find the solution of the differential equation

$$\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{1+x^2}$$

with initial condition $y(0) = 0$.

Separating variables here gives $\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{1+x^2}$.
Solving this gives $\arcsin y = \arctan x + C$ and substituting $y(0) = 0$ we find $C = 0$.
Therefore $y = \sin(\arctan x) = \frac{x}{\sqrt{1+x^2}}$.

(a) $y = \frac{x}{1+x}$ (b) $y = \frac{1}{\sqrt{1+x^2}}$ (c) $y = \frac{x}{\sqrt{1+x^2}}$

(d) $y = \frac{x^2}{\sqrt{1+x^2}}$ (e) $y = \frac{x}{1+x^2}$

10.(7 pts.) Find a general solution, valid for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, of the differential equation

$$\frac{dy}{dx} - (\tan x)y = 1.$$

The integrating factor here is $I = e^{\int -\tan x dx} = e^{\ln(\cos x)} = \cos x$ and so a general solution is given by $y \cos x = \int \cos x dx = \sin x + C$. Therefore $y = \frac{\sin x + C}{\cos x}$.

(a) $y = \frac{x + \sin x + C}{\cos x}$ (b) $y = \tan x + \cos x + C$ (c) $y = \frac{\cos x + C}{\sin x}$

(d) $y = \frac{\sin x + C}{\cos x}$ (e) $y = \frac{x + \sin x + C}{\sin x}$

Name: ______________________________

Instructor: __________________________

11.(7 pts.) Which of the following statements about the sequence

$$\left\{\frac{(\ln n)^2}{n}\right\}_{n=1}^{\infty}$$

is true?

Solution: Use L'Hospital's rule (twice):

$$\lim_{x\to\infty}\frac{(\ln x)^2}{x}=\lim_{x\to\infty}\frac{((\ln x)^2)'}{x'}=\lim_{n\to\infty}\frac{2\frac{1}{x}\ln x}{1}=\lim_{x\to\infty}\frac{(2\ln x)'}{x'}=\lim_{x\to\infty}\frac{\frac{2}{x}}{1}=0.$$

The sequence converges to 0.

(a) The sequence converges to e^2

(b) The sequence converges to ∞

(c) The sequence converges to 0

(d) The sequence diverges

(e) The sequence converges to 1

12.(7 pts.) Find the family of orthogonal trajectories to the family of curves given by

$$y = kx^2.$$

$$\frac{dy}{dx} = 2kx$$

For the family of curves given above

$$y = kx^2 \;\; \text{giving} \;\; k = \frac{y}{x^2}$$

Thus this family of curves satisfy the differential equation

$$\frac{dy}{dx} = 2\frac{y}{x^2}x = 2\frac{y}{x}.$$

Name: ______________________________

Instructor: __________________________

Now using the fact that the product of the derivatives of two orthogonal curves meeting at a point must equal -1, we get that the orthogonal trajectories satisfy the differential equation

$$\frac{dy}{dx} = \frac{-x}{2y}.$$

Separating the variables, we get

$$2ydy = -xdx$$

and

$$2\int ydy = -\int xdx, \quad \text{or} \quad y^2 = \frac{-x^2}{2} + C.$$

Hence our family of orthogonal trajectories is a family of curves of the form

$$y^2 + \frac{x^2}{2} = C,$$

a family of ellipses.

(a) $y = x^2 + C$

(b) $y = Cx^2$

(c) $y^2 - \frac{x^2}{2} = C$

(d) $y^2 + \frac{x^2}{2} = C$

(e) $y^2 - x^2 = C$

Name: __________________________

Instructor: ________________________

Partial Credit

You must show your work on the partial credit problems to receive credit!

13.(10 pts.) Calculate the arc length of the curve if $y = \frac{x^2}{4} - \ln(\sqrt{x})$, where $2 \leq x \leq 4$.

Solution: Recall

$$L = \int_a^b \sqrt{1 + (y')^2} dx.$$

Note

$$y' = \frac{x}{2} - \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2}\Big(x - \frac{1}{x}\Big).$$

Thus

$$1 + (y')^2 = 1 + \frac{1}{4}\Big(x - \frac{1}{x}\Big)^2 = 1 + \frac{1}{4}\Big(x^2 - 2x\frac{1}{x} + \frac{1}{x^2}\Big) = 1 + \frac{1}{4}\Big(x^2 - 2 + \frac{1}{x^2}\Big)$$
$$= 1 + \frac{1}{4}x^2 - \frac{1}{2} + \frac{1}{4x^2} = \frac{1}{4}x^2 + \frac{1}{2} + \frac{1}{4x^2} = \frac{1}{4}\Big(x^2 + 2x\frac{1}{x} + \frac{1}{x^2}\Big) = \frac{1}{4}\Big(x + \frac{1}{x}\Big)^2.$$

Therefore

$$L = \int_2^4 \sqrt{1/4(x + 1/x)^2} dx = \int_2^4 \frac{1}{2}\Big(x + \frac{1}{x}\Big) dx = \frac{1}{2}\left[\frac{x^2}{2} + \ln x\right]_2^4 = 3 + \frac{1}{2}\ln 2.$$

Name: ______________________________

Instructor: ___________________________

14.(10 pts.) Solve the initial value problem

$$xy' + xy + y = e^{-x}$$
$$y(1) = \frac{2}{e}.$$

Solution: This is a linear differential equation. Since it can be reduced to the form

$$y' + \left(1 + \frac{1}{x}\right)y = \frac{e^{-x}}{x},$$

an integrating factor is

$$I(x) = e^{\int (1+\frac{1}{x})dx} = e^{x+\ln x} = xe^{x}.$$

Multiply both sides of the differential equation by $I(x)$ to get

$$xe^{x}y' + y(x+1)e^{x} = 1,$$

and hence

$$(xe^{x}y)' = 1.$$

Integrate both sides to obtain

$$xe^{x}y = x + C,$$

or

$$y = e^{-x}\left(1 + \frac{C}{x}\right).$$

Using the initial value, we have

$$y(1) = \frac{2}{e} = \frac{1}{e}(1 + C), \qquad C = 1.$$

Hence

$$y = e^{-x}\left(1 + \frac{1}{x}\right).$$

Name: ______________________

Instructor: ______________________

15. (6 pts.) Please circle "TRUE" if you think the statement is true, and circle "FALSE" if you think the statement is False.

(a)(1 pt. No Partial credit) The formula for the trapezoidal approximation with n approximating trapezoids is given by

$$\frac{\Delta x}{2}\left[2f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_n)\right].$$

TRUE FALSE

(b)(1 pt. No Partial credit) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ converges.

TRUE FALSE

(c))(1 pt. No Partial credit) $\displaystyle\int_1^{\infty} \frac{1}{x}$ diverges.

TRUE FALSE

(d))(1 pt. No Partial credit) If $\dfrac{dy}{dx} = \sqrt{x^2 + y^2}$ is a separable differential equation.

TRUE FALSE

(e))(1 pt. No Partial credit) The sequence $\left\{1 - \dfrac{1}{n}\right\}_{n=1}^{\infty}$ diverges.

TRUE FALSE

(f))(1 pt. No Partial credit) The slope field for the differential equation $\dfrac{dy}{dx} = xe^{-y}$ has a line with slope -1 at the point $(-1, 0)$.

TRUE FALSE

Name: ______________________________

Instructor: __________________________

The following is the list of useful trigonometric formulas:

Note: $\sin^{-1} x$ and $\arcsin(x)$ are different names for the same function and $\tan^{-1} x$ and $\arctan(x)$ are different names for the same function.

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin 2x = 2 \sin x \cos x$$

$$\sin x \cos y = \frac{1}{2}\big(\sin(x - y) + \sin(x + y)\big)$$

$$\sin x \sin y = \frac{1}{2}\big(\cos(x - y) - \cos(x + y)\big)$$

$$\cos x \cos y = \frac{1}{2}\big(\cos(x - y) + \cos(x + y)\big)$$

$$\int \sec\theta = \ln|\sec\theta + \tan\theta| + C$$

$$\int \csc\theta = \ln|\csc\theta - \cot\theta| + C$$

$$\csc\theta = \frac{1}{\sin\theta}, \quad \cot\theta = \frac{1}{\tan\theta}$$